Fundamental problems of QCD

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I. Quantum Chromodynamics: formulation

- Non-abelian gauge theory (Yang - Mills, 1954).
- Quark model (Gell-Mann, Zweig, 1964).
- Discovery of new quantum number of quarks called colour (Bogolyubov-Struminski-Tavkhelidze; Nambu-Khan, 1965).
- Perturbative quantization of nonabelian gauge models (Faddeev-Popov, 1967).
- Proof of the renormalization of non-abelian models ('t Hooft, 1969-1971).
- Lagrangian of QCD (Fritzsch-Gell-Mann-Leutwyler, 1973).
- Discovery of asymptotic freedom (Politzer, Gross-Wilczek, 1973).
- Non-perturbative quantization of gauge models and proof of confinement in the strong coupling regime (Wilson, 1974).

Quantum numbers and masses of quark flavours

q	B	J	Q	m (MeV)
u	1/3	1/2	2/3	2.3
d	1/3	1/2	-1/3	4.8
s	1/3	1/2	-1/3	95
c	1/3	1/2	2/3	1275
b	1/3	1/2	-1/3	4180
t	1/3	1/2	2/3	173210

B is the baryon number, J is spin, Q - electric charge and m - quark mass. Examples: proton $p = uud$, neutron $n = udd$, $\Delta^{++} = uuu$.

Quark flavours: u - up, d - down, s - strange, c - charm, b - beauty, t - top.

Quarks can be in 3 colour states: red, green, blue.

Gauge (gluon) fields are massless at the classical level.

Lagrangian which possesses local gauge symmetry and describes interaction between N_f species of fermions has the following form in the continuum

$$\mathcal{L} \;=\; -\frac{1}{g^2}\,\mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + \sum_{f=1}^{N_f} \;\overline{\Psi}^f \;[iD_\mu\gamma_\mu - m_f)]\;\Psi^f \;,$$

where the strength tensor of the gauge field is

$$F_{\mu\nu} = \partial_\mu A_\nu \;-\; \partial_\nu A_\mu + i[A_\mu, A_\nu]\;,\; A_\mu = A_\mu^a t^a\;.$$

and the covariant derivative reads

$$D_\mu = \partial_\mu - iA_\mu\;.$$

$\mu = 1, ..., d$, where d is space-time dimension. $t^a \in A(G)$ are generators of an algebra $A(G)$ of some gauge group G, $a = 1, ..., N_c$. m_f is bare fermion mass and N_f is a number of fermion species. g^2 is the bare coupling constant. The action takes the form

$$S(A, \overline{\Psi}^f, \Psi^f) \;=\; \int d^dx\;\mathcal{L}$$

The theory which describes the strong interaction of quarks is called Quantum Chromodynamics, or in short, QCD.

Main properties

1. QCD Lagrangian is invariant under local transformations. If $U(x) \in G$
$$\mathcal{L}(A_\mu, \overline{\Psi}^f, \Psi^f) \; = \; \mathcal{L}(A'_{mu}, \overline{\Psi}^{\prime,f}, \Psi^{\prime,f}) \; ,$$
$$A'_\mu = U(x) A_\mu U^+(x) - i[\partial_\mu U(x)] U^+(x) \; ,$$
$$\overline{\Psi}^{\prime,f} = \overline{\Psi}^f U^+(x) \; , \; \Psi^{\prime,f} = U(x) \Psi^f \; .$$

2. Elitzur theorem: Gauge symmetry cannot be spontaneously broken.

3. When $m_f = 0$, QCD Lagrangian is invariant under global chiral transformations $U(N_f) \times U(N_f)$.

4. Let $Z \in Z(3)$ be center element of $SU(3)$
$$Z \; = \; \exp\left[i\lambda_8 \, \omega_8\right] \; = \; \exp\left[\frac{2\pi i k}{3}\right] \; I \; , \; \omega_8 = \frac{2\pi k}{\sqrt{3}} \; , \; k = 0, 1, 2 \; .$$
Then, gauge fields are invariant under global $Z(3)$ transformations. Quark fields transform like above.

Non-perturbative quantization

The action which has been proposed by Wilson for $SU(N)$ LGT is defined in terms of the plaquette variables

$$S[U] \;=\; \beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{N}\mathrm{Re\ Tr}U_{\mu\nu}(x)\right) \;,$$

where the sum extends over all plaquettes of the lattice and β is the inverse bare coupling constant. Gauge field matrices $U_\mu(x) \in G$ and are taken in the fundamental representation of the group and

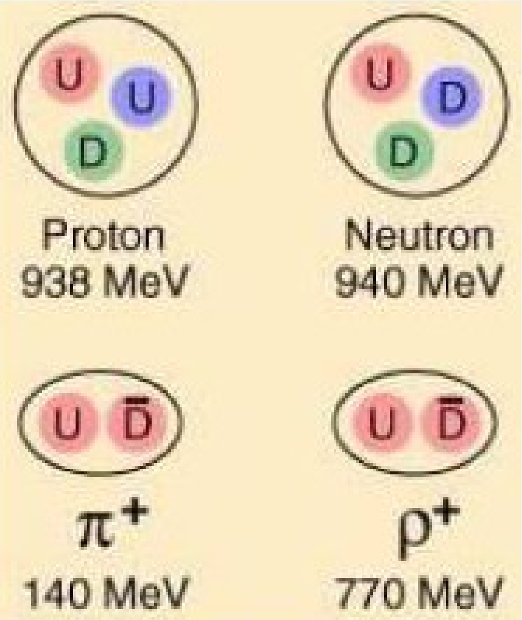
$$U_{\mu\nu}(x) \;=\; U_\mu(x)U_\nu(x+\widehat{\mu})U_\mu^\dagger(x+\widehat{\nu})U_\nu^\dagger(x) \;.$$

Quantum theory is given by the sum over all gauge field configurations

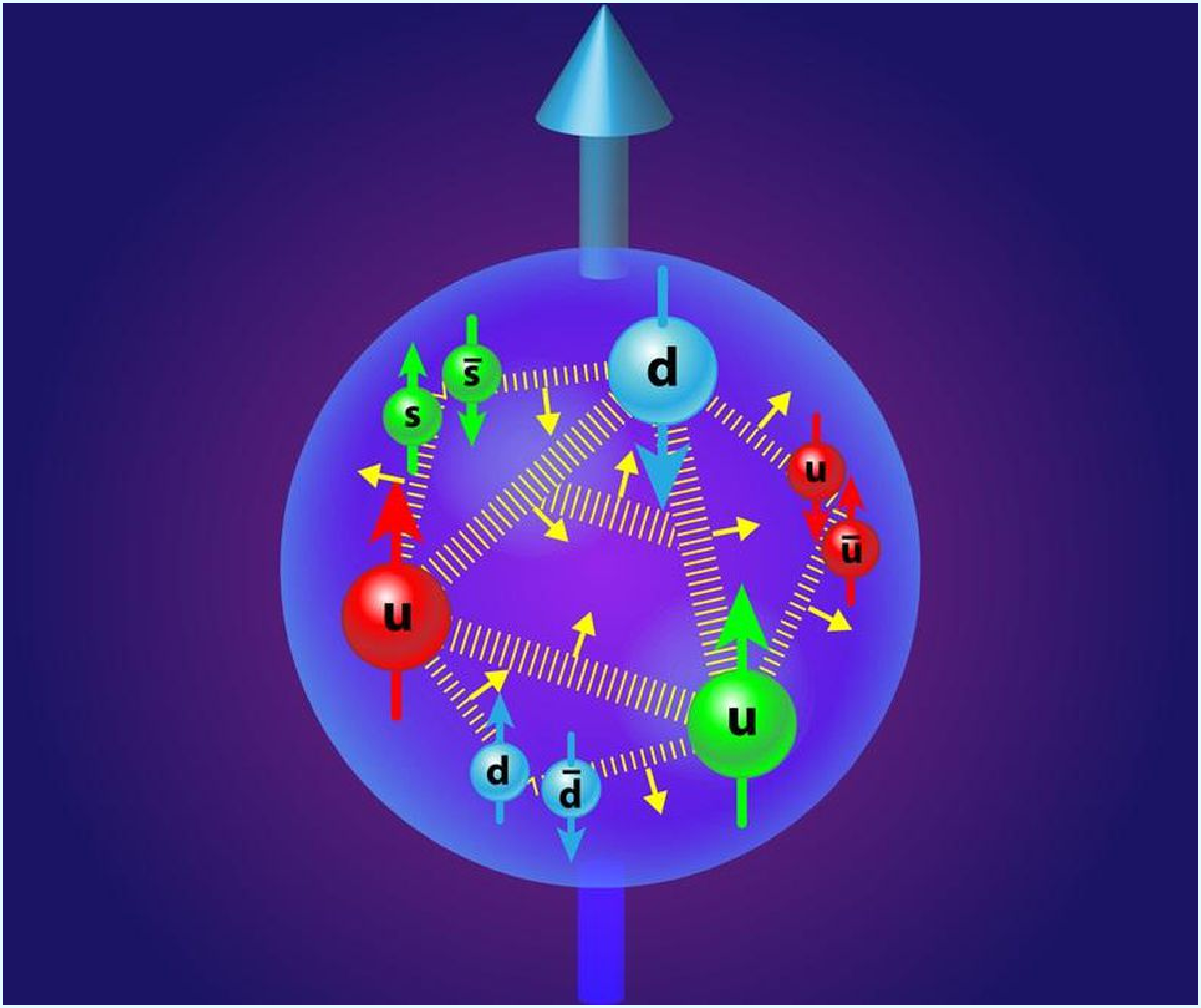
$$Z \;=\; \int \prod_{x,\mu} dU_\mu(x) \; \exp\left(-S[U]\right) \;.$$

II. Strong interactions

Observable particles are not quarks but hadrons: baryons (bound state of 3 quarks) and mesons (bound state of quark–antiquark pair). QCD must describe 1) how quarks are bound into hadrons and 2) how hadrons interact with each other.

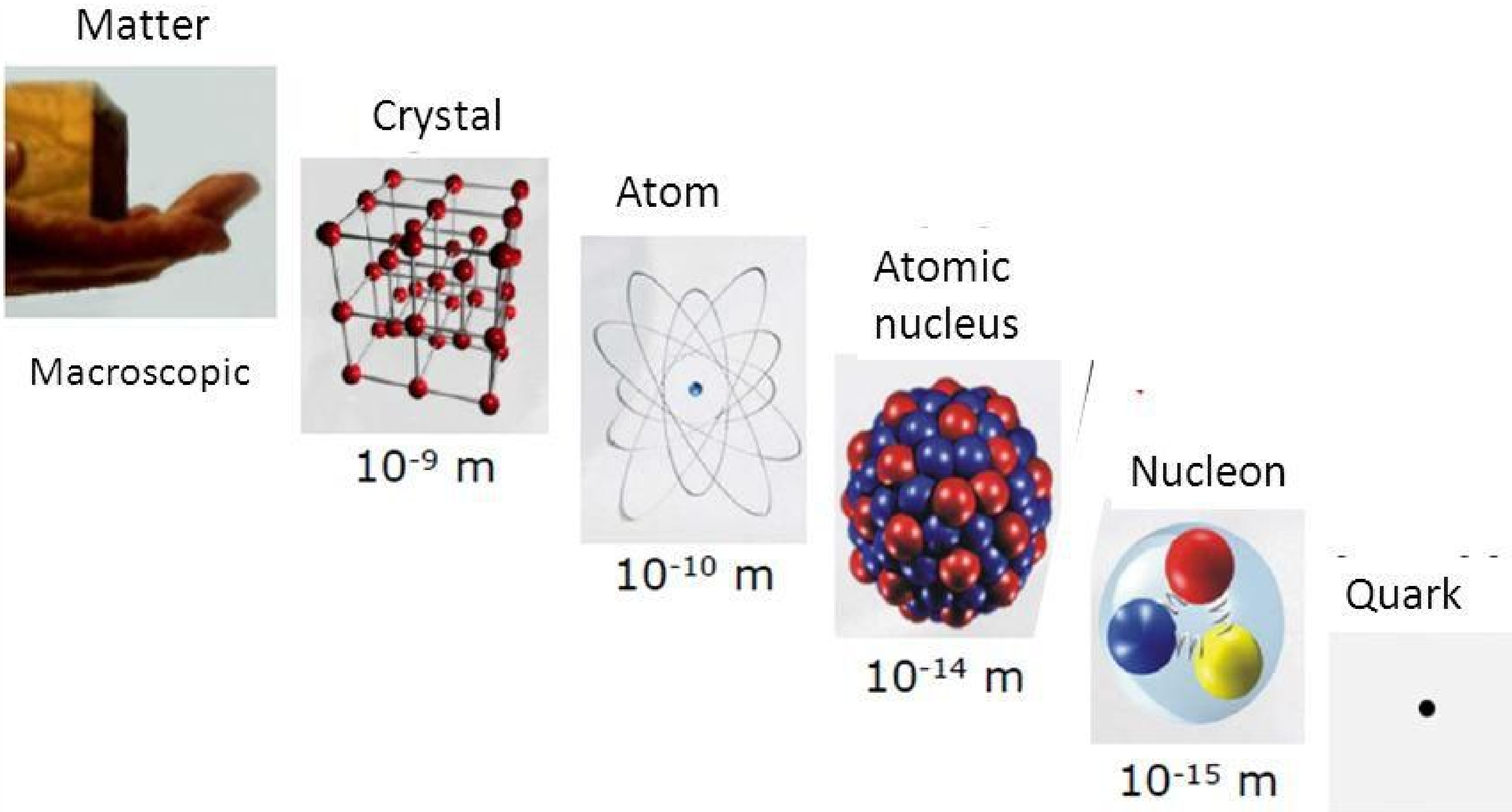


Naive quark model of mesons and baryons



Proton in QCD

Nuclear scale



Nuclear physics:
studies the properties of nuclei and
the interactions inside and between them

III. Quantum Chromodynamics: problems of the strong interactions

For QCD to successfully describe the strong force, it must have at the quantum level the following three properties, each of which is dramatically different from the behaviour of the classical theory:

- **the nuclear force is strong but short-ranged**
  $\rightarrow$ **QCD must have mass gap**

- **we have to explain quark masses**
  $\rightarrow$ **QCD must have chiral symmetry breaking**

- **we never see individual quarks**
  $\rightarrow$ **QCD must have quark confinement**

There does not exist a convincing, whether or not mathematically complete theoretical computation demonstrating any of the three properties (mass gap, confinement, chiral symmetry breaking) in QCD, as opposed to a severely simplified truncation of it.

Classical properties of gauge theory are within the reach of established mathematical methods. On the other hand, one does not yet have a mathematically complete example of a quantum gauge theory in four-dimensional space-time, nor even a precise definition of quantum gauge theory in four-dimensions.

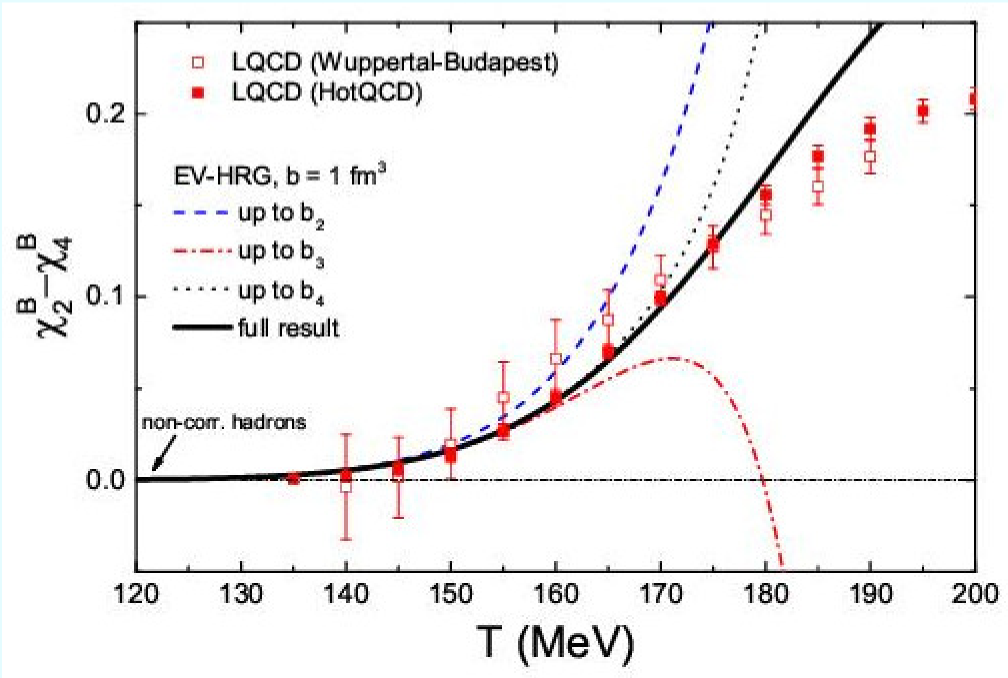
IV. Short-range nature of the strong interactions

Quarks inside hadrons interact very strongly. After a limiting distance (about the size of a hadron) has been reached, it remains at a strength of about 10,000 newtons , no matter how much farther the distance between the quarks. The strong force is much weaker between colourless objects like neutrons and protons: the potential decreases exponentially fast.

How one could explain this property? The main conjecture is that QCD has a mass gap. Gluons are massless at the classical level. In reality, gluons cannot exist as coloured states: they are confined and can form bound states called glueballs. Glueballs are massive composite particles.

Hadrons cannot interact via exchange of massless gluons but only via massive glueballs and light mesons, therefore the force falls rapidly with distance between hadrons.

Example: description of lattice data for the baryonic susceptibilities by the hadron resonance gas (HRG) model (ideal gas of hadrons)



HRG describes perfectly well lattice data for most thermodynamical quantities up to the deconfinement phase transition point $\approx$ 160 MeV.

The only theory confining electric charges at all values of the bare coupling and for which the generation of the mass gap was proven is three-dimensional $U(1)$ gauge theory. Mechanism of the confinement and the mass gap generation appears to be tightly connected.

$$Z \;=\; \sum_{m_x} \exp\left[-\pi^2\beta\sum_{x,y} m_x G_{x,y}(0) m_y\right] \;.$$

$G_{x,y}(0)$ is massless photon Green function. Contribution of the monopole configurations $\{m_x\}$ gives rise both to non-zero string tension and to mass of the gauge field: $G_{x,y}(0)$ (and so photon) becomes massive.

How this mass gap generation agrees with the perturbative expansion which relies on the existence of the continuous spectrum and massless gauge fields? The mass gap is exponentially small in the bare coupling, $m \sim \mathbf{e}^{-\beta}$. Therefore, PT, as an expansion in $1/\beta$, cannot feel it by its very virtue.

Gluons in QCD are massless but they do not appear in the spectrum. What can appear in the spectrum are massive colourless states called glueballs. Glueball masses in QCD are calculated from the exponential decay of the connected part of the (properly modified) plaquette-plaquette correlation function

$$\left\langle \mathrm{Tr} U(p)\, U(p') \right\rangle_c \sim e^{-m|x-y|} .$$

Exact form of this correlation depends on the quantum numbers of glueball J^{PC}. Numerical results for some lightest glueball masses

0^{++}	1709 Mev
2^{++}	2388 Mev
0^{-+}	2557 Mev

The problem of the mass gap generation is summarized in

The Millenium problem: Quantum Yang-Mills Theory

A. Jaffe, E. Witten

Yang-Mills Existence and Mass Gap: Prove that for any compact simple gauge group G, a non-trivial quantum Yang-Mills theory exists on R^4 and has a mass gap $\triangle > 0$.

V. Mass of visible matter in the Universe

The whole is equal to the sum of its parts. This rule is true about almost everything in the Universe. If you were to break a human being down into our constituent components, the cells in our body would add up to our entire selves. Same for the molecules in our cells and the atoms in our molecules. But when we get down to atomic nuclei, something funny happens: the individual protons and neutrons are about 1 percent heavier than the atoms as a whole. That is a clue as to what's happening, but it cannot prepare us for the most mind-boggling fact: the quarks that make up the proton are only 0.2 percents of the protons actual mass!

$m_u = 2.3$ Mev, $m_d = 4.8$ Mev, $m_p = 938$ Mev

Where the difference come from?

Approximately 99 percents of the mass of visible matter in the Universe is due to protons and neutrons.

Chiral symmetry of QCD

Consider the following transformations of the quark fields

$$U(N_f) \times U(N_f) \; = \; SU_L(N_f) \times SU_R(N_f) \times U_V(1) \times U_A(1)$$

Chiral rotations act on $\Psi^f_{L,R} = \frac{1 \pm \gamma_5}{2} \Psi^f$ and leave kinetic part of the QCD Lagrangian invariant. The mass term breaks this symmetry explicitely. However, as masses of u and d quarks are very small, this explicit breaking can be neglected in the first approximation in the theory with two or even three lightest flavours.

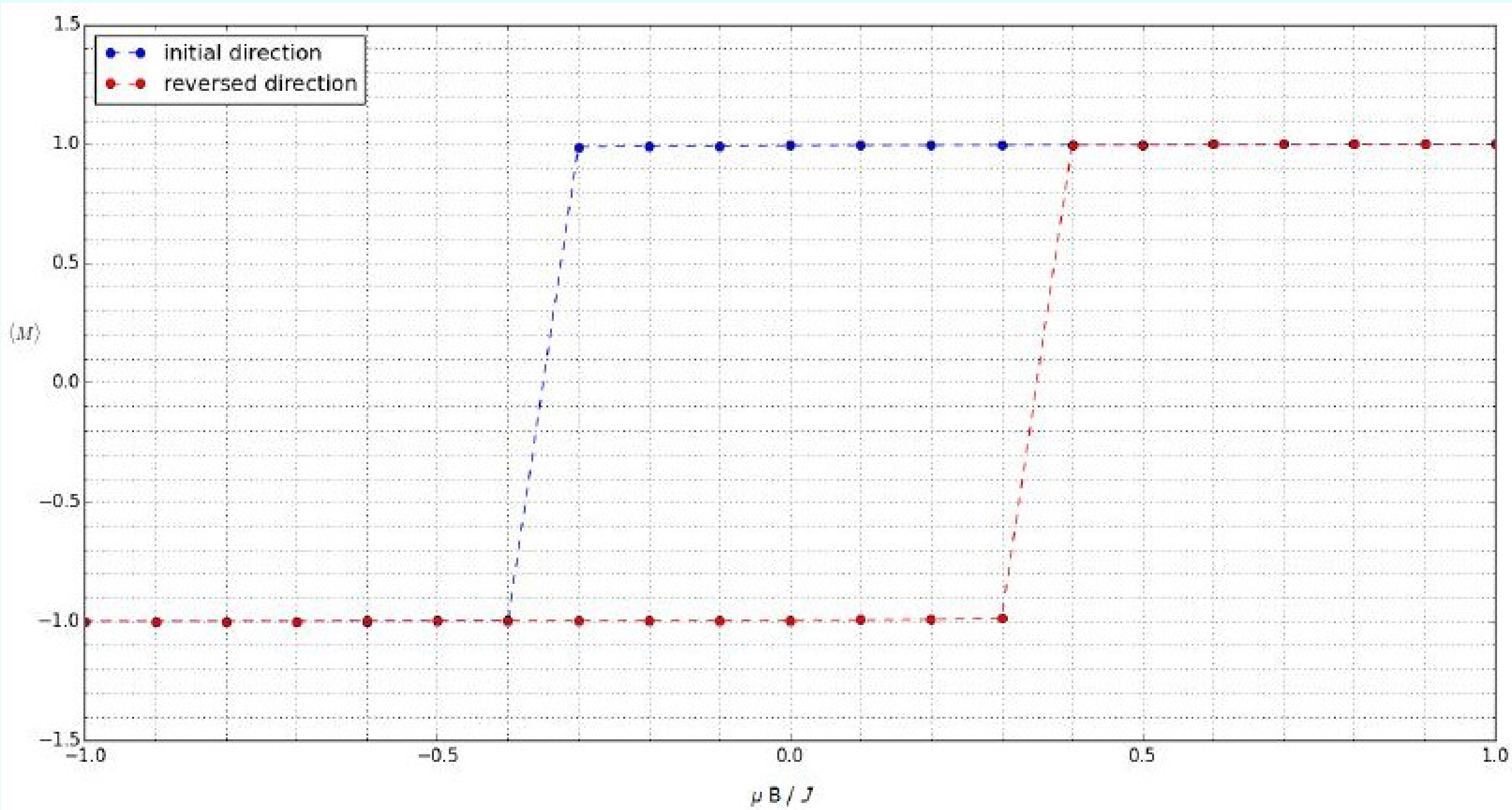
The main conjecture is that QCD has a spontaneous chiral symmetry breaking.

The order parameter of this breaking is called quark condensate

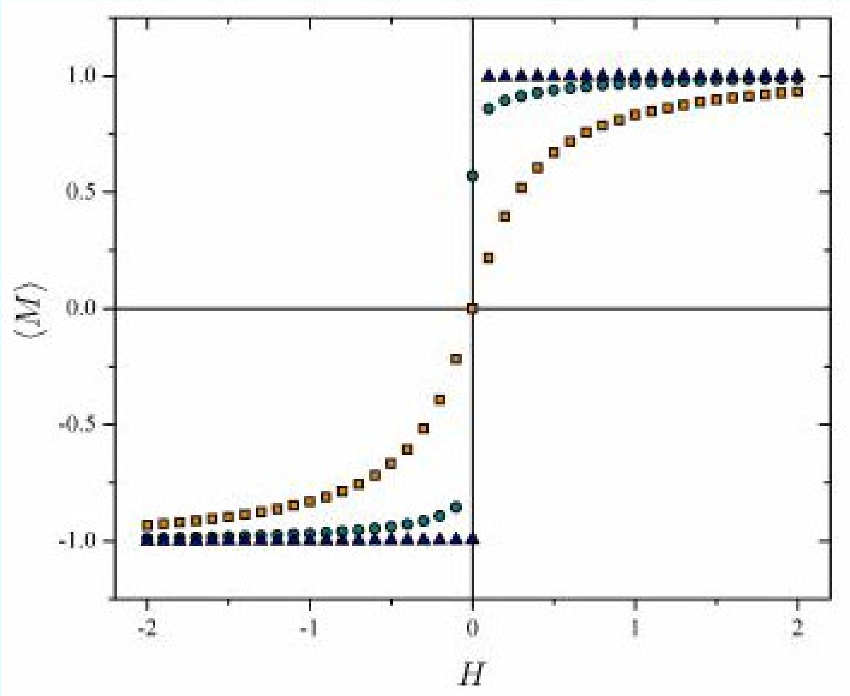
$$\sigma \; = \; \sum_{i=1}^{N_c} \bar{\Psi}^f_i \Psi^f_i$$

If $\sigma \neq 0$, the resulting effective theory of hadron bound states of QCD has a mass term both for mesons and for baryons. Unfortunately, such effective theory can be reliably computed only in the strong coupling limit.

The spontaneous symmetry breaking may be described in analogy to magnetization. As example: two-dimensional Ising model in the external magnetic field



Average magnetization $< M >$ vs external magnetic field B for $\beta = 0.75$ (the critical value $\beta_c \approx 0.44$, $B = 0$).



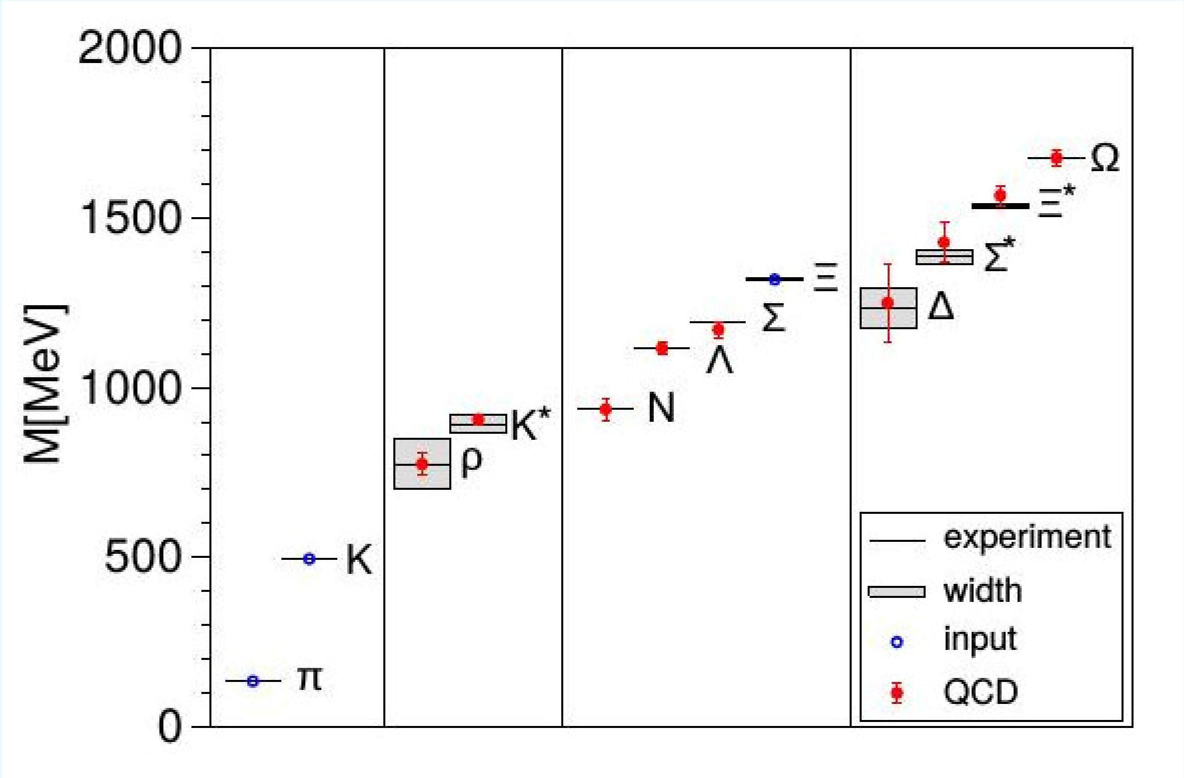
Average magnetization $< M >$ vs external magnetic field H for $\beta = 0.769$ (triangles), $\beta = 0.4407$ (circles), $\beta = 0.303$ (squares). The critical value $\beta_c \approx 0.44$, $B = 0$.

Hadron masses can be computed from the two-point correlation functions:

$$\langle O(t_1, x)\, O(t_2, x)\rangle_c \sim \sum_n C_n\, e^{-m_n|t_1 - t_2|} .$$

Operators $O(t, x)$ are composed of quark fields Ψ_f^a, γ and group matrices to form colourless state with desirable quantum numbers and symmetry properties. Even if such operators are local (not the case for realistic hadrons) thanks to universality their correlations should behave as exact hadron correlations near the continuum limit.

Hadron masses obtained in lattice QCD by Budapest-Wupertal group (2009)



The light hadron spectrum of QCD. π, K, Ξ are used to set masses of u, d, s quarks.

VI. Why we do not see the most fundamental particles

Mass gap generation in QCD is not a unique phenomenon. One encounters similar ones in a variety of systems ranging from classical spin models to many other QFTs.

Spontaneous breaking of the symmetry takes place also in a number of models (magnetization in Ising model).

What is unique in QCD is the absence of the fundamental particles - quarks - in the spectrum of the theory.

It was conjectured (by whom?) that there is a permanent confinement of quarks, so that these elementary particles cannot be observed in principle.

Confinement is a distinguished feature of QCD: nothing similar we can find in any other physical theory.

Confinement and triality

QCD Lagrangian is invariant under global center transformations. The center elements commute with all other group elements and can be written for $SU(N)$ group as

$$Z \; = \; \exp\left[\frac{2\pi i k}{N}\right] \; I \; , \; k = 0, 1, ..., N-1 \; .$$

The gauge fields are identically invariant and quark fields in the fundamental representation transform nontrivially as

$$\overline{\Psi}^{\prime,f} = Z^{*} \, \overline{\Psi}^{f} \; , \; \Psi^{\prime,f} = Z \, \Psi^{f} \; .$$

If the quark field is taken in a representation trivial on the center subgroup then it is invariant under the center transformation. Accordingly to these transformation rules a quantum number N-ality (triality for QCD) is assigned to gauge and quark fields. Gauge fields have N-ality zero, quarks carry a unit of N-ality charge and antiquarks have N-ality (-1). N-ality is a multiplicative quantum number, i.e. it is defined up to modulo N (in QCD $N = 0$ is equivalent to $N = 3, 6, ...$). All experimentally observed particles have vanishing N-ality. Mesons are composed by quark-antiquark pair, thus N-ality is zero. Baryons consist of N quarks or N antiquarks. Hence, their N-ality equals $\pm N$ which is equivalent to zero. In $U(1)$ gauge theory the center coincides with $U(1)$ group.

The confinement problem can now be formulated as follows:

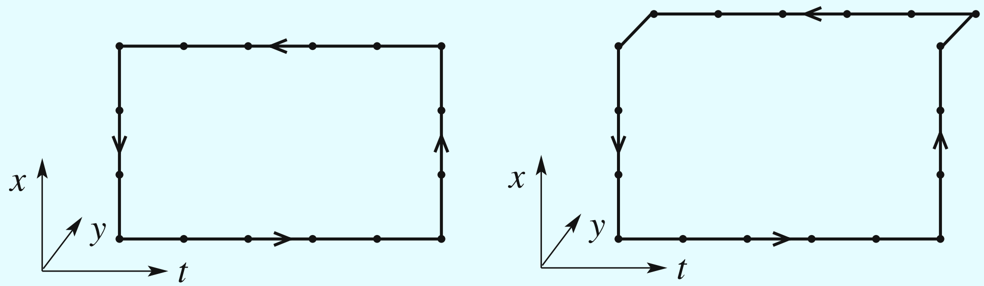
starting from nonperturbative regularization of QCD construct a proof that all physical states, predicted by QCD, have vanishing triality.

In other words, one has to prove that all asymptotic states with nonvanishing triality cannot exist.

The next question is how to translate this physically obvious question to a mathematical question expressed in terms of quantities natural for QCD. This is rather nontrivial question since QCD Lagrangian is not written in terms of mesons and baryons but in terms of fields which carry colour charge. The answer to this question is well known for the pure gauge models.

Wilson loop and $q\bar{q}$ (meson) potential

Wilson loop describes creation of static (very heavy) quark–anti-quark pair at time T_1 separated by distance R, propagation of the pair in time direction and its annihilation at time T_2. Examples are:



Planar (left) and simplest non-planar (right) Wilson loops.

Expectation value of the Wilson loops is directly related to the potential between quark–anti-quark pair (meson potential) or to potential between three quarks (baryon potential).

Confinement, Wilson loop and string tension

$$W(C) \;=\; \langle\, \mathsf{Tr} \prod_{\mathsf{l} \in \mathsf{C}} \mathsf{U_l} \,\rangle \;\sim\; \mathsf{exp}(-\mathsf{F_{q\bar{q}}})\;,$$

$F_{q\bar{q}}$ - free energy of a quark–anti-quark pair.

Confinement: Prove that 1) the string tension between static quark–anti-quark pair

$$\sigma_{q\bar{q}}(\beta) \;=\; -\lim_{S(C)\to\infty} \frac{1}{S(C)} \;\mathsf{ln}\; W(C)\;.$$

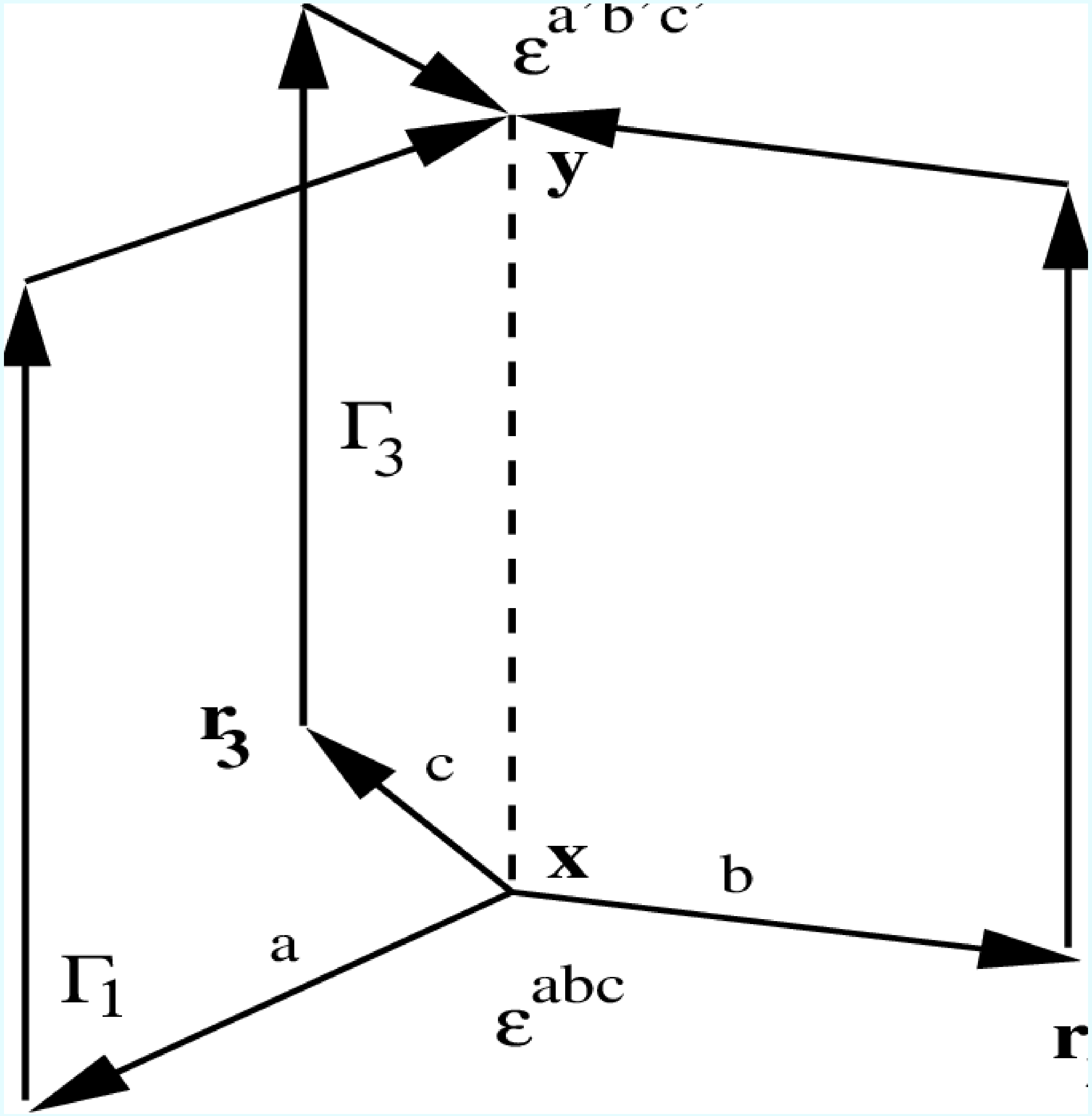
is non-vanishing at all values of the bare coupling β and 2) continuum limit of σ exists. $S(C)$ - area enclosed by the loop C. Equivalently: prove area law decay of the Wilson loop.

For a restangular loop $S = RT$, the potential between quark–anti-quark pair is

$$V_{q\bar{q}} \;=\; -\lim_{T\to\infty} \frac{1}{T} \;\mathsf{ln}\; W(C) \;=\; \sigma_{q\bar{q}}\, R\;.$$

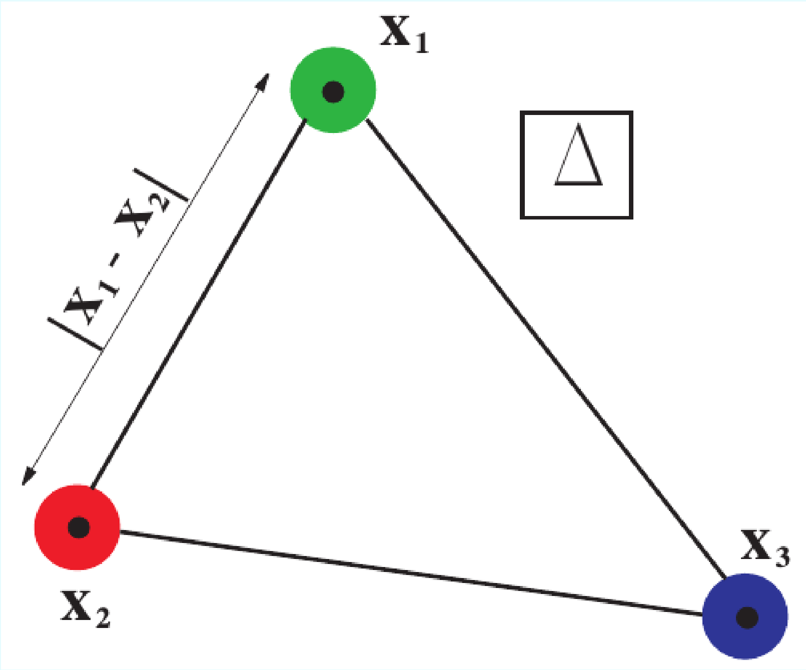
If $W(C)$ obeys area law decay, the potential grows linearly with distance R. Therefore: Anti-quark cannot be removed to infinity and state with non-zero triality (quark) does not exist (its free energy is infinite).

The baryonic Wilson loop (book observable) is used to calculate qqq potential.



$3q$ (baryon) potential: $\triangle$ law

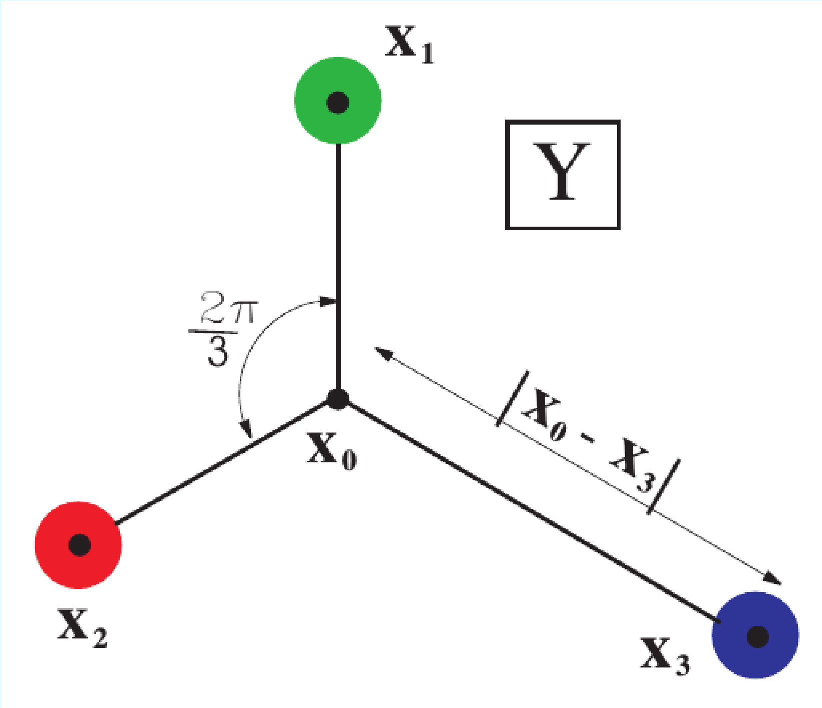
$3q$ potential is calculated from baryonic Wilson loop (aslo called book observable). Two different laws are possible.



$\triangle$ law: $3q$ potential is a sum of two-body linear potentials between each pair:

$$V_{\triangle} = \sigma_{qqq} \triangle , \; \triangle = \frac{1}{2} \sum_{i<j} |x_i - x_j| \, .$$

$3q$ (baryon) potential: Y law



$$V_Y = \sigma_{qqq}\, Y \;,\; Y = \mathrm{min}_{x_0} \sum_{i=1}^{3} |x_i - x_0| \;,$$

Y law: $3q$ potentail is a genuine three-body potential, where x_0 is the Fermat-Torricelli point.

In all cases it is found that $\sigma_{q\bar{q}} = \sigma_{qqq}$.

Strong coupling expansion and most recent Monte-Carlo simulations support Y law but the issue is still under debates.

String tension behaviour

Abelian models, $d > 2$

Three-dimensional model confines (electric) charges in all representations and at all values of the coupling constant (permanent confinement)

$$W_j(C) = \langle\, e^{ij\phi(C)} \,\rangle \;\sim\; \exp(-\sigma_j(\beta)\, S(C)) \; .$$

At small and intermediate distances the Casimir scaling dominates

$$\sigma_j(\beta) \;\sim\; j^2\, \sigma_1(\beta) \; .$$

Exact lower bound on $\sigma_1(\beta)$ is known. At large distances sandwitch type diagrammes give leading contribution

$$\sigma_j(\beta) \;\sim\; j\, \sigma_1(\beta) \; .$$

Four-dimensional model confines charges at strong couplings. Here the string tension behaves like in $3d$ theory. At $\beta \approx 1$ phase transition takes place. Above critical point Wilson loop has perimeter decay and electric charges are free.

Non-Abelian models, $d > 2$

On every link belonging to the loop C one encounters integral of the type

$$\int dU\, U^j \prod_{p \in l} U^{r(p)}$$

Non-vanishing result arises only if the product of matrix elements in the integrand contains trivial representation. This simple rule leads to appearing four different types of diagrammes: Casimir diagram, sandwich diagram, N-ality diagram and perimeter one (plus corrections).

- Fundamental string tension is known to at least 14th order in β
- The strong coupling behaviour described below agrees with Monte-Carlo simulations in the scaling region

1. Wilson loops non-trivial on the center at small and intermediate distances obeys area law. Leading diagram is of Casimir type:

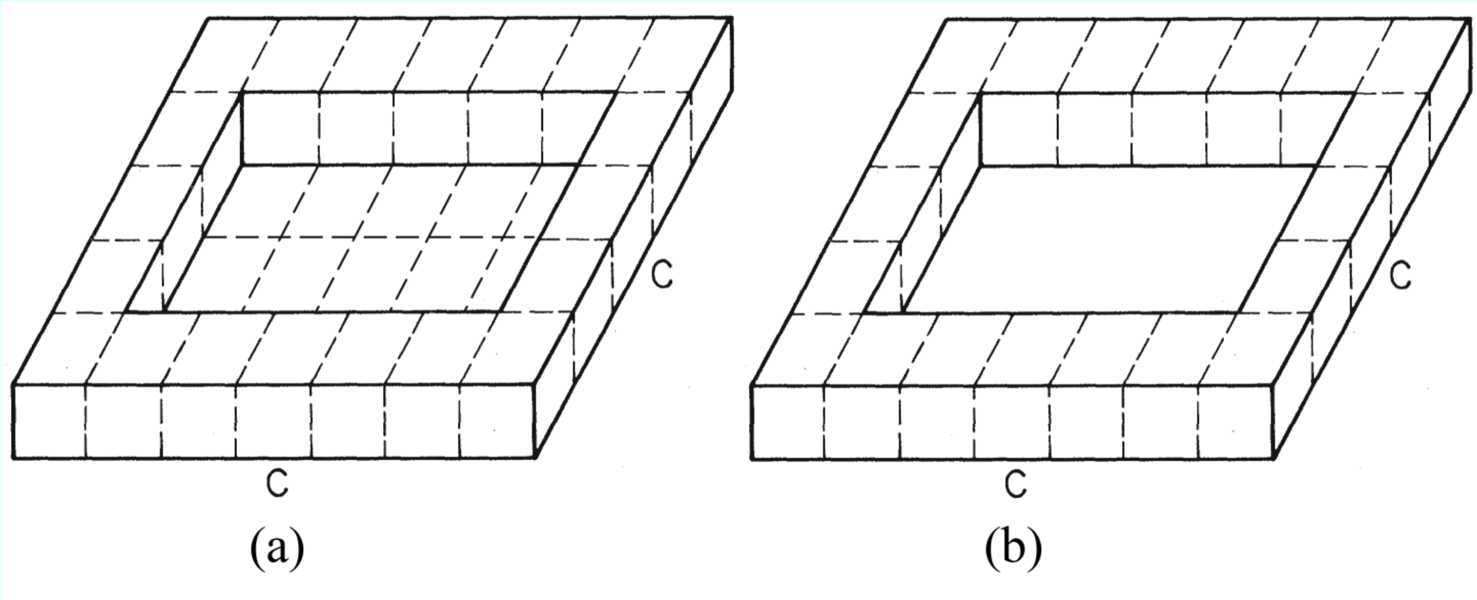
$$\sigma_j(\beta) \sim C_2(N;\{j\}) \ .$$

2. Wilson loops non-trivial on the center at large distances obeys area law. Leading diagram is of N-ality type. It means the string tension does not depend on representation and equals the fundamental string tension

$$\sigma_j(\beta) \sim \sigma_f(\beta) \ .$$

3. Wilson loops trivial on the center at small and intermediate distances obeys area law. Leading diagram is of Casimir type.

4. Wilson loops trivial on the center at large distances obeys perimeter law, i.e. the corresponding charges are not confined and can be separated.



(a) is N-ality diagram exhibiting independence of the asymptotic string tension of the quark representation if it is non-trivial on the center. Area enclosed by the loop C taken in a representation $\{j\}$ is covered once by plaquettes from the action in the fundamental representation. The rest is compensated by the perimeter contribution.

(b) describes perimeter law (absence of confinement) for representations trivial on the center.

Mechanism of quark confinement

- What are gauge field configurations which provide area law decay of the fundamental Wilson loop? More generally: which are responsible for expected behaviour of Wilson loops in all representations and on all scales?

- What is physics of confinement?

- What is continuum limit of such configurations, if any? Do they have any relevance for experiment?

- Confinement in two-dimensional theories
- Confinement in $U(1)$ gauge theory and monopoles
- Confinement in $Z(N)$ gauge theory and vortices
- Monopole and vortex mechanisms in non-abelian models

VII. Summary

QCD explains, though not rigorously, the most important features of the strong interactions. It explains the mass of the visible matter in the Universe. A number of open questions remains, the most important are:

- Mechanism of confinement and mass gap generation
- $\triangle$ vs Y law; Confinement in theories with matter fields
- Relation between confinement and chiral symmetry breaking
- How to compute effective baryon-meson theory from first principles
- When we solve all these problems?

Analytically, NEVER

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